University of California, Berkeley
Physics 110A Spring 2003 Section 2 (*Strovink*)

SOLUTION TO MIDTERM EXAMINATION I

Directions: Do all three problems, which have unequal weight. This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. A photocopy of the four inside covers of Griffiths is included with the exam. Calculators are not needed, but you may use one if you wish. Laptops and palmtops should be turned off. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Show all your work. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Express your answer in terms of the quantities specified in the problem. Box or circle your answer.

Problem 1. (35 points)
A volume charge density $\rho(\vec{r})$ has the value over all space

$$\rho(\vec{r}) = \sigma_0 \delta(s-b) \ ,$$

where σ_0 and b are positive constants, and s is the usual cylindrical coordinate (the perpendicular distance from the z axis). (Remember that the usual integral formula for getting V from ρ can be used only if $\rho = 0$ at infinity!)

(**a**) (5 points) What are the dimensions of σ_0?
Solution:
Since $\delta(s-b)$ has dimensions m^{-1} (over s it integrates to unity), and ρ has dimensions $\mathrm{C/m^3}$, σ_0 has dimensions $\mathrm{C/m^2}$ – it is a surface charge density.

(**b**) (15 points) Find the electric field $\vec{E}(\vec{r})$ at any space point $\vec{r} = (s, \phi, z)$.
Solution:
The charge giving rise to $\vec{E}$ is distributed in an infinite cylindrical shell of radius b centered on the z axis. Because of the cylindrical symmetry and z-independence we know that $\vec{E}$ must point along $\hat{s}$ and be independent of ϕ or z. As a Gaussian surface we choose a cylindrical can of radius s and length L, also centered on the z axis. Applying Gauss's Law,

$$\begin{aligned}
\oiint \epsilon_0 \vec{E} \cdot d\vec{a} &= Q \\
2\pi s L \epsilon_0 E_s &= 2\pi b L \sigma_0 \quad (s > b) \\
&= 0 \quad (s \le b) \\
\epsilon_0 \vec{E} &= \hat{s}\, \sigma_0 \frac{b}{s} \quad (s > b) \\
&= 0 \quad (s \le b) \ .
\end{aligned}$$

(**c**) (10 points) Find the potential difference

$$\Delta V \equiv V(b,0,0) - V(0,0,\infty) \ ,$$

where, as above, the parentheses refer to (s, ϕ, z).
Solution:
Both sets of coordinates lie within the field-free region $s \le b$, so $\Delta V = 0$.

Problem 2. (35 points)
The potential energy of mutual electrostatic interaction between two ideal electric dipoles $\vec{p}_1$ and $\vec{p}_2$ is

$$4\pi\epsilon_0 U_{12} = -p_1 p_2 \frac{3(\hat{r}\cdot\hat{p}_1)(\hat{r}\cdot\hat{p}_2) - \hat{p}_1\cdot\hat{p}_2}{r^3} \ ,$$

where $\vec{r}$ is their mutual separation. Consider a single ideal dipole with electric dipole moment equal in magnitude to p. The dipole is a distance z above an infinite grounded conducting plane $z = 0$.

(**a**) (10 points) Assume for this part that the dipole is allowed to rotate so that it may point in any direction. Does it tend to point perpendicular to the plane (*i.e.* along $\hat{z}$), or parallel to the plane? Your answer should be justified by either a quantitative or a qualitative argument.
Solution:
The true and mirror dipoles are always separated such that $\hat{r} = \hat{z}$ (the sign of $\hat{r}$ doesn't matter). Suppose the true dipole $\vec{p}_1$ is oriented vertically ($\hat{p}_1 = \hat{z}$). Consider the mirror dipole. The vector $\vec{d}$ from the negative to the positive

mirror charge points in the same absolute direction as $\vec{p}_1$. Therefore, for this orientation of $\hat{p}_1$, the mirror dipole moment is the *same* as the true dipole moment: $\vec{p}_2 = +\vec{p}_1$. Then

$$\frac{4\pi\epsilon_0 (2z)^3 U_{12}}{p_1 p_2} = -(3-1) = -2 \ .$$

Suppose instead that the true dipole p_1 is oriented horizontally ($\hat{p}_1 =$ (say) $\hat{x}$). Then the mirror dipole moment is *opposite* to the true dipole moment, and

$$\frac{4\pi\epsilon_0 (2z)^3 U_{12}}{p_1 p_2} = -(0-(-1)) = -1 \ .$$

In the first case U_{12} is more negative, so the dipole tends to be oriented *perpendicular* to the plane.

(**b**) (10 points) Assume instead that the dipole's direction is fixed so that it points perpendicular to the plane. Is the plane attracted to or repelled from the dipole? Again your answer should be justified by either a quantitative or a qualitative argument.

Solution:

From part (**a**), regardless of dipole orientation, U_{12} becomes more negative as z diminishes. Therefore the true dipole is attracted downward to the conducting plane, and correspondingly the conducting plane is attracted upward to the true dipole.

(**c**) (15 points) For the conditions of part (**b**), calculate the magnitude of this attractive or repulsive force.

Solution:

U_{12} is the electrostatic field energy associated with the interaction of two *real* dipoles. Here only the electrostatic field energy above the grounded plane – half of the total electrostatic field energy represented by U_{12} – is real. Therefore the attractive force is only *half* the negative gradient of U_{12}: on the true dipole

$$\begin{aligned} F_z &= -\tfrac{1}{2}\frac{dU_{12}}{dz} \\ &= -\tfrac{1}{2}\frac{d}{dz}(-2)\frac{p^2}{4\pi\epsilon_0 (2z)^3} \\ &= -\frac{3p^2}{32\pi\epsilon_0 z^4} \ . \end{aligned}$$

[Very substantial part credit is awarded for twice the correct answer.]

Problem 3. (30 points)

A conducting sphere of radius b, centered at the origin, is surrounded by a spherical insulating layer of material with dielectric constant $\epsilon_r \equiv \epsilon/\epsilon_0$, extending from $b < r < 2b$. The conductor has a spherical hole of radius $b/4$ centered at $(x, y, z) = (0, 0, b/4)$. At the center of the hole is a charge q. There is no other *net* charge on any material. Take θ to be the usual spherical polar angle between $\hat{z}$ and $\hat{r}$.

(**a**) (10 points) Find the free charge density $\sigma_f(\theta)$ on the outside surface of the conducting sphere.

Solution:

Statically there can be no electric field inside the bulk conducting material of which the sphere is composed. Consider a spherical surface centered on the hole, a small distance inside the bulk material that bounds the hole. Applying Gauss's law to this surface, it must contain no charge. Therefore a charge $-q$ must be distributed on the inside surface of the hole. Because the sphere has no net charge, a cancelling charge $+q$ must therefore be distributed on its outside surface. Because the sphere's outside surface is an equipotential, the electric field outside it must also be spherically symmetric; thus the cancelling charge $+q$ must be distributed uniformly over this surface. Therefore

$$\sigma_f = \frac{q}{4\pi b^2} \ .$$

(**b**) (20 points) Find the potential $V(\theta)$ at $r = b$, assuming that $V = 0$ at $r = \infty$.

Solution:

Consider a spherical surface outside the conductor, centered at the origin. Taking advantage of the spherical symmetry of the fields outside the conductor, and applying Gauss's law for $\vec{D}$, one obtains

$$4\pi\vec{D} = \hat{r}\frac{q}{r^2} \ .$$

Then, using the linearity of the dielectric,

$$\begin{aligned} 4\pi\epsilon_0\vec{E} &= \hat{r}\frac{q}{r^2} \quad (2b < r) \\ &= \hat{r}\frac{q}{\epsilon_r r^2} \quad (b < r \le 2b) \ . \end{aligned}$$

Integrating,

$$
\begin{aligned}
4\pi\epsilon_0\big(V(b) - V(\infty)\big) &= -4\pi\epsilon_0 \int_\infty^b \vec{E}\cdot d\vec{l} \\
&= 4\pi\epsilon_0 \int_b^\infty E_r\, dr \\
&= \int_b^{2b} \frac{q}{\epsilon_r r^2} dr + \int_{2b}^\infty \frac{q}{r^2} dr \\
&= \frac{q}{\epsilon_r b} - \frac{q}{2\epsilon_r b} + \frac{q}{2b} \\
&= \frac{q}{2b}\big(1 + \frac{1}{\epsilon_r}\big)\ .
\end{aligned}
$$